JOURNAL OF APPROXIMATION THEORY **38**, 240–257 (1983)

Interpolation of 2^d Banach Spaces and Multiparametric Approximation

DICESAR LASS FERNANDEZ

Instituto de Matemática, Universidade Estadual de Campinas, 13.100—Campinas—S.P., Brazil

Communicated by P. L. Butzer

Received January 5, 1982; revised May 11, 1982

A multiparametric approximation theory via certain inequalities of Jackson and Bernstein type is developed. An approximation space is defined and it is shown that it is actually an interpolation space among 2^d Banach spaces. As applications, direct and converse approximation theorems in function spaces with a dominant mixed derivative are given.

INTRODUCTION

The close connection existing between classical approximation theory and the theory of interpolation spaces is well known. The possibility of applying interpolation techniques to approximation theory was first indicated by Peetre [16]. In that paper Peetre gave an abstract approximation theory via certain approximation spaces and showed that these spaces are actually interpolation spaces. He also gave applications to the approximation of functions in Sobolev spaces by entire functions of exponential type. Since then, the theory of interpolation spaces has been applied in approximation theory by several authors. (See for instance Butzer [4], Berg and Löfström [3] and the references quoted in these works.)

The study of interpolation spaces has hitherto been restricted mainly to couples of Banach spaces. However, real methods of interpolation for several Banach spaces, in the sense of Lions and Peetre [10] and Peetre [16], have been studied by Johnen [9], Yoshikawa [18] and Sparr [17]. Johnen [9] not only introduced an interpolation theory but also gave applications to approximation theory. All of these authors concerned themselves with $d + 1$ spaces and d parameters. On the other hand, Fernandez [5] has introduced an interpolation theory for 2^d Banach spaces and d parameters. This approach is useful for application to, e.g., multiparametric approximation theory.

Following Peetre [16] we shall here formulate a d-parametric approximation theory in normed spaces. This will be done, as in [16], via certain

0021-9045/83 $3.00

approximation spaces which are actually interpolation spaces of 2^d Banach spaces. As applications we derive direct and converse theorems on the approximation of functions in function spaces with a dominant mixed derivative, by entire functions of exponential type. These theorems permit us to recover as a consequence some results by Nikol'skii (see Nikol'skii [13], Lizorkin and Nikol'skii [12] and Amanov [1]).

1. Interpolation of 2^d Banach Spaces

We shall here give a summary of facts on the theory of interpolation of 2^d Banach spaces that shall be needed in the following. For the proofs see Fernandez [5].

1.1. *Generalities on Interpolation for 2^d Banach Spaces*

1.1.1. The set of $k=(k_1,..., k_d) \in \mathbb{R}^d$ such that $k_j=0$ or 1 will be denoted by $\square$. We have $\square=\{0, 1\}$ when $d=1$, and $\square=\{(0,0), (1,0), (0,1), (1,1)\}$ when $d=2$. The families of objects we shall consider will depend on indices in $\square$.

1.1.2. We shall consider families of 2^d Banach spaces $\mathbb{E}=(E_k \mid k \in \square)$ embedded, algebraically and continuously, in one and the same linear Hausdorff space V. Such a family will be called an admissible family of Banach spaces (in V).

1.1.3. If $\mathbb{E}=(E_k \mid k \in \square)$ is an admissible family of Banach spaces, the linear hull $\Sigma\mathbb{E}$ and the intersection $\cap\mathbb{E}$ are defined in the usual way. They are Banach spaces under the norms

$$(1) \qquad \|x\|_{\Sigma\mathbb{E}}=\inf\{\Sigma_k \|x_k\|_{E_k} \mid x=\Sigma_k x_k ;\, x_k \in E_k,\, k \in \square\}$$

and

$$(2) \qquad \|x\|_{\cap\mathbb{E}}=\max\{\|x\|_{E_k} \mid k \in \square\}.$$

The spaces $\cap\mathbb{E}$ and $\Sigma\mathbb{E}$ are continuously embedded in V.

1.1.4. A Banach space E which satisfies

$$(1) \qquad \cap\mathbb{E} \subset E \subset \Sigma\mathbb{E}$$

will be called an intermediate space (with respect to $\mathbb{E}$). (Hereafter $\subset$ will denote a continuous embedding).

1.2. *The Intermediate Spaces* $(E_k \mid k \in \square)_{\Theta;Q;K}$

1.2.1. Let $\mathbb{E} = (E_k \mid k \in \square)$ be an admissible family of Banach spaces; for $x \in \Sigma\mathbb{E}$ and $t = (t_1, ..., t_d) > 0$ we set

$$(1) \quad K(t;x) = K(t;x;\mathbb{E}) = \inf\{\Sigma_k t^k \|x_k\|_{E_k} \mid x = \Sigma_k x_k, x_k \in E_k, k \in \square\}$$

(as usual $t^k = t_1^{k_1} \cdots t_d^{k_d}$).

Now, assume $0 < \Theta = (\theta_1, ..., \theta_d) < 1$ and $1 \leqslant Q = (q_1, ..., q_d) \leqslant \infty$.

1.2.2. Definition. We define $(E_k \mid k \in \square)_{\Theta;Q;K}$ to be the space of all elements $x \in \Sigma\mathbb{E}$ for which

$$(1) \qquad t^{-\Theta} K(t;x) \in L_*^Q.$$

Here L_*^Q stands for the L^Q spaces with mixed norms of Benedek and Panzone [2] with respect to the measure $d_* t = dt/t = dt_1/t_1 \cdots dt_d/t_d$.

1.2.3. Proposition. *The spaces* $(E_k \mid k \in \square)_{\Theta;Q;K}$ *are Banach spaces under the norms*

$$(1) \qquad \|x\|_{\Theta;Q;K} = \|t^{-\Theta} K(t;x)\| L_*^Q.$$

Furthermore, the spaces $(E_k \mid k \in \square)_{\Theta;Q;K}$ *are intermediate spaces with respect to* $\mathbb{E}$, *i.e.*,

$$(2) \qquad \cap\mathbb{E} \subset (E_k \mid k \in \square)_{\Theta;Q;K} \subset \Sigma\mathbb{E}.$$

1.3. *The Intermediate Spaces* $(E_k \mid k \in \square)_{\Theta;Q;J}$

1.3.1. Let $\mathbb{E} = (E_k \mid k \in \square)$ be an admissible family of Banach spaces. For $x \in \cap\mathbb{E}$ and $t = (t_1, ..., t_d) > 0$ we get

$$(1) \qquad J(t;x) = J(t;x;\mathbb{E}) = \max\{t^k \|x\|_{E_k} \mid k \in \square\}.$$

Again, assume $0 < \Theta = (\theta_1, ..., \theta_d) < 1$ and $1 \leqslant Q = (q_1, ..., q_d) \leqslant \infty$.

1.3.2. Definition. We define $(E_k \mid k \in \square)_{\Theta;Q;J}$ to be the space of all elements $x \in \Sigma\mathbb{E}$ for which there exists a strongly measurable function $u: \mathbb{R}_+^n \to \cap\mathbb{E}$ such that

$$(1) \qquad x = \int_{\mathbb{R}_+^d} u(t) d_* t \qquad (\text{in } \Sigma\mathbb{E}),$$

and

$$(2) \qquad t^{-\Theta} J(t; u(t)) \in L_*^Q.$$

1.3.3. PROPOSITION. *The spaces* $(E_k \mid k \in \square)_{\Theta;Q;J}$ *are Banach spaces under the norms*

$$(1) \qquad \|x\|_{\Theta;Q;J} = \inf \left\{ \|t^{-\Theta} J(t; u(t))\|_{L_*^Q} \mid x = \int_{\mathbb{R}_+^d} u(t) d_* t \right\}.$$

Furthermore, the spaces $(E_k \mid k \in \square)_{\Theta;Q;J}$ *are intermediate with respect to* $\mathbb{E}$, *e.g.,*

$$(2) \qquad \cap\mathbb{E} \subset (E_k \mid k \in \square)_{\Theta;Q;J} \subset \Sigma\mathbb{E}.$$

We shall say the spaces $(E_k \mid k \in \square)_{\Theta;Q;K}$ and $(E_k \mid k \in \square)_{\Theta;Q;J}$ are generated by the K- and the J-methods, respectively.

1.4. *The Identity between* $(E_k \mid k \in \square)_{\Theta;Q;K}$ *and* $(E_k \mid k \in \square)_{\Theta;Q;J}$

The following result gives a connection between the spaces generated by the K- and the J-method, and states that these methods are actually equivalent.

1.4.1. PROPOSITION. *If* $0 < \Theta = (\theta_1, ..., \theta_d) < 1$ *and* $1 \leqslant Q = (q_1, ..., q_d) \leqslant \infty$ *we have*

$$(1) \qquad (E_k \mid k \in \square)_{\Theta;Q;J} = (E_k \mid k \in \square)_{\Theta;Q;K}.$$

When we have no need to specify which interpolation method has generated the intermediate space we shall write simply $(E_k \mid k \in \square)_{\Theta;Q}$.

1.5. *The Reiteration Theorem*

One of the central results in the theory of interpolation spaces is the reiteration or stability theorem. In order to state it we need some preliminaries.

1.5.1. DEFINITION. Let $0 < \Theta = (\theta_1, ..., \theta_d) < 1$. We say an intermediate space E with respect to $\mathbb{E} = (E_k \mid k \in \square)$ belongs to the class

$$(1) \qquad K(\Theta; \mathbb{E}) \quad \text{iff} \quad K(t; x; \mathbb{E}) \leqslant C t^{\Theta} \|x\|_E \qquad (x \in E);$$

$$(2) \qquad J(\Theta; \mathbb{E}) \quad \text{iff} \quad \|x\|_E \leqslant C t^{-\Theta} J(t; x; \mathbb{E}) \qquad (x \in \cap\mathbb{E}).$$

1.5.2. PROPOSITION. *We have*

$$(1) \qquad E \in K(\Theta; \mathbb{E}) \quad \textit{iff} \quad E \subset (E_k \mid k \in \square)_{\Theta;\infty;K};$$

$$(2) \qquad E \in J(\Theta; \mathbb{E}) \quad \textit{iff} \quad (E_k \mid k \in \square)_{\Theta;1;J} \subset E.$$

Now we can state the reiteration theorem.

1.5.3. PROPOSITION. *Given an admissible family* $\mathbb{E} = (E_k \mid k \in \square)$ *and a family of parameters* $(\Theta_k = (\theta^1_{k_1},..., \theta^d_{k_d}) \mid k = (k_1,..., k_d) \in \square)$, *let us consider a family of intermediate spaces such that* $F_k \in K(\Theta; \mathbb{E}) \cap J(\Theta; \mathbb{E})$. *Thus, if* $0 < \Theta = (\theta_1,..., \theta_d) < 1$ *and* $\Lambda = (\lambda_1,..., \lambda_d)$ *is defined by* $\lambda_j = (1-\theta_j)\theta^j_0 + \theta_j\theta^j_1$, $j = 1, 2,..., d$, *we have*

(1) $$(F_k \mid k \in \square)_{\Theta;Q} = (E_k \mid k \in \square)_{\Lambda;Q}.$$

1.6. *The Interpolation Property*

The interpolation property holds for the intermediate spaces $(E_k \mid k \in \square)_{\Theta;Q}$.

1.6.1. PROPOSITION. *Let* $\mathbb{E} = (E_k \mid k \in \square)$ *and* $\mathbb{F} = (F_k \mid k \in \square)$ *be two families of admissible Banach spaces in* V *and* W, *respectively. If* T *is a linear mapping from* $\Sigma\mathbb{E}$ *into* $\Sigma\mathbb{F}$ *such that*

(1) $$T \mid E_k : E_k \to F_k, \qquad k \in \square,$$

we have

(2) $$T: (E_k \mid k \in \square)_{\Theta;Q} \to (F_k \mid k \in \square)_{\Theta;Q}.$$

(*Arrows will always denote bounded linear mappings.*)

Remark. In view of the above result we shall hereafter call the space $(E_k \mid k \in \square)_{\Theta;Q}$ an *interpolation space.*

2. THE SPACES OF SOBOLEV–NIKOL'SKIĬ AND BESOV–NIKOL'SKIĬ

2.1. *The Sobolev–Nikol'skiĭ and the Besov–Nikol'skiĭ Spaces*

We shall here recall the definition and some properties of some function spaces introduced by Nikol'skiĭ. (See, e.g., Nikol'skiĭ [13], Lizorkin–Nikol'skiĭ [12], Amanov [1].)

Throughout this article we shall be dealing with locally summable functions on $\mathbb{R}^d$. The derivatives are always taken in the weak sense (see Nikol'skiĭ [14, pp. 141–151]). As before, the spaces $L^P = L^P(\mathbb{R}^d)$ are the L^P spaces with mixed norms of Benedek and Panzone [2].

2.1.1. Let there be given a fixed multi-index $M = (m_1,..., m_d) \in \mathbb{N}^d$ and $1 \leqslant P = (p_1,..., p_d) \leqslant \infty$. We define the Sobolev–Nikol'skiĭ space $W^{M,P}$ by

(1) $$W^{M,P} = W^{M,P}(\mathbb{R}^d) = \{u \in L^P \mid D^\alpha u \in L^P, \alpha \leqslant M\}.$$

(Recall that $\alpha = (\alpha_1,..., \alpha_d) \leqslant M = (m_1,..., m_d)$ iff $\alpha_j \leqslant m_j$, $j = 1,..., d$.)

The spaces $W^{M,P}$ are complete under the norm

(2) $$\|u\|_{W^{M,P}}=\|u\|_{M,P}=\sum_{\alpha\leqslant M}\|D^\alpha u\|_{L^P}.$$

2.1.2. For a function f from $\mathbb{R}^d$ into $\mathbb{R}$ we define the *mixed difference* of order $M=(m_1,\ldots,m_d)\in\mathbb{N}^d$ by

(1) $$\begin{aligned}\Delta_h^M f(x)&=\sum_{0\leqslant J\leqslant M}(-1)^{M-J}\binom{M}{J}f(x+J(h))\\&=\sum_{j_d=0}^{m_d}\cdots\sum_{j_1=0}^{m_1}(-1)^{m_1-j_1}\cdots(-1)^{m_d-j_d}\binom{m_1}{j_1}\\&\quad\cdots\binom{m_d}{j_d}f(x_1+j_1h_1,\ldots,x_d+j_dh_d),\end{aligned}$$

and *the mixed moduli of continuity* of order $M=(m_1,\ldots,m_d)\in\mathbb{N}^d$ by

(2) $$\omega_M(t;f)=\omega_{m_1,\ldots,m_d}(t_1,\ldots,t_d;f)=\sup_{\substack{|h_j|\leqslant t_j\\ j=1,\ldots,d}}\|\Delta_h^M f\|_{L^P}.$$

Besides the mixed difference $\Delta_h^M f$ and the mixed moduli of continuity $\omega_M(t;f)$ we shall deal with the *partial differences* $\Delta_h^{k\circ M}f$ and the *partial moduli of continuity* $\omega_{k\circ M}(t;f)$, $k\in\square$. (Hereafter we set $k\circ M=(k_1m_1,\ldots,k_dm_d)$.)

Now, the Besov–Nikol'skiĭ spaces can be introduced.

2.1.3. Let there be given $S>0$ and $M\in\mathbb{N}^d$ such that

$$0<S=(s_1,\ldots,s_d)<M=(m_1,\ldots,m_d)$$

and

$$1\leqslant P=(p_1,\ldots,p_d),\quad Q=(q_1,\ldots,q_d)\leqslant\infty.$$

We define $B_{M,P}^{S,Q}=B_{M,P}^{S,Q}(\mathbb{R}^d)$ to be the space of all $f\in L^P(\mathbb{R}^d)$ such that

(1) $$t_1^{-k_1s_1}\cdots t_d^{-k_ds_d}\omega_{k_1m_1\cdots k_dm_d}(t_1,\ldots,t_d;f)=t^{-k\circ S}\omega_{k\circ M}(t;f)\in L_*^Q$$

for all $k=(k_1,\ldots,k_d)\in\square$.

The spaces $B_{M,P}^{S,Q}$ are complete under the norm

(2) $$\|f\|_{B_{M,P}^{S,Q}}=\sum_{k\in\square}\|t^{-k\circ S}\omega_{k\circ M}(t;f)\|_{L_*^Q}.$$

2.2. *Interpolation of the Sobolev–Nikol'skiĭ Spaces*

We shall now give a characterization of the Besov–Nikol'skiĭ spaces as interpolation spaces among 2^n Sobolev–Nikol'skiĭ spaces. For the proofs see Fernandez [6].

2.2.1. If $f \in L^P$, let us set

$$\Omega_M(t; f) = \sum_{k \in \square} (\min t^k)\, \omega_{k \circ M}(t; f). \tag{1}$$

2.2.2. PROPOSITION. *Let $K(t; f)$ be the interpolation functional associated with the admissible family $(W^{k \circ M, P} \mid k \in \square)$. Then*

$$K(t; f) \simeq \Omega_M(t^{1/M}; f) = \Omega_{m_1 \cdots m_d}(t_1^{1/m_1}, ..., t_d^{1/m_d}; f). \tag{1}$$

2.2.3. COROLLARY. *If $0 < S = (s_1, ..., s_d) < M = (m_1, ..., m_d)$ and $1 \leqslant P = (p_1, ..., p_d)$, $Q = (q_1, ..., q_d) \leqslant \infty$, we shall have*

$$(W^{k \circ M, P} \mid k \in \square)_{S/M; Q; K} = B^{S, Q}_{M, P}. \tag{1}$$

Now, for the reiteration theorem we need the following result.

2.2.4. LEMMA. *If $0 < N = (n_1, ..., n_d) < M = (m_1, ..., m_d)$ the space $W^{N, P}$ belongs to both classes $K(N/M;\ (W^{k \circ M, P} \mid k \in \square))$ and $J(N/M;\ (W^{k \circ M, P} \mid k \in \square))$.*

As a consequence of this lemma we obtain the reiteration theorem.

2.2.5. THEOREM. *If $0 < N = (n_1, ..., n_d) \leqslant M - 1 = (m_1 - 1, ..., m_d - 1)$ and $0 < \Theta = (\theta_1, ..., \theta_d) < 1$, we have*

$$(W^{N+k} \mid k \in \square)_{\Theta; Q} = (W^{k \circ M} \mid k \in \square)_{(N + \Theta)/M; Q} = B^{N+\Theta; Q}_{M, P}. \tag{1}$$

2.2.6. *Remark.* As a consequence of the reiteration theorem we see, as in the case of the usual Besov spaces (see Peetre [15]), that the Besov–Nokol'skĭi spaces $B^{S,Q}_{M,P}$ do not depend on the parameter M. Thus, we shall simply write $B^{S,Q}_P$.

We close this section by stating the reduction theorem.

2.2.7. PROPOSITION. *Let $0 < S < M$ and if $0 < \beta < 1$ let N be such that $S = N + \beta$ and $0 \leqslant N \leqslant M - 1$. If $f \in L^P(\mathbb{R}^d)$ the following statements are equivalent:*

$$f \in B^{S,Q}_P = B^{S,Q}_{M,P}; \tag{1}$$

$$f \in W^{N,P} \quad \text{and} \quad D^{k \circ N} f \in B^{\beta, Q}_{1,P} = B^{\beta, Q}_P, \quad \text{all } k \in \square. \tag{2}$$

3. A THEORY OF APPROXIMATION IN NORMED SPACES

We shall give here a theory of approximation in normed spaces. We shall define two approximation spaces via some inequalities of Jackson and Bernstein type.

3.1. *The Approximation Space* $E_{\alpha;Q;K}$

3.1.1. Let E be a Banach space and let us consider a multiple scale $(W_M \mid M \in \mathbb{N}^d)$ of subspaces of E, i.e.,

(1) $$W_0 = \{0\},$$

(2) $$W_{M'} \subset W_{M''} \quad \text{if} \quad M' \leqslant M''.$$

3.1.2. For every $M \in \mathbb{N}^d$ and every $x \in E$ we introduce the best approximation of x by elements of W_M by

(1) $$\mathscr{E}_M(x) = \inf\{\|x - w\|_E \mid w \in W_M\}.$$

It follows that

(2) $$\mathscr{E}_0(x) = \|x\|_E$$

and

(3) $$\mathscr{E}_{M'}(x) \geqslant \mathscr{E}_{M''}(x) \quad \text{if} \quad M' \leqslant M''.$$

3.1.3. Definition. Let $0 < \alpha = (\alpha_1, ..., \alpha_d) \leqslant \infty$ and $1 \leqslant Q = (q_1, ..., q_d) \leqslant \infty$. We define $E_{\alpha;Q;K}$ to be the space of all $x \in E$ such that

(1) $$\|x\|_{\alpha;Q;K} = \|(e^{\alpha \cdot M} \mathscr{E}_M(x))_{M \in \mathbb{N}^d}\|_{l^Q(\mathbb{N}^d)} < \infty.$$

(Recall that $\alpha \cdot M = \alpha_1 m_1 + \cdots + \alpha_d m_d$).

The spaces $E_{\alpha;Q;K}$ are Banach spaces under the norms 3.1.3(1).

The following result follows at once.

3.1.4. Proposition. *If* $x \in E_{\alpha;\infty;K}$ *we have*

$$\mathscr{E}_M(x) \leqslant e^{-\alpha \cdot M} \|x\|_{\alpha;\infty;K}.$$

The inequality 3.1.4(1) is an inequality of Jackson type and this permits us to call $E_{\alpha;Q;K}$ an *approximation space.*

3.2. *The Approximation Space* $E_{\alpha;Q;J}$

As before, let E be a Banach space and $(W_M \mid M \in \mathbb{N}^d)$ a multiple scale of subspaces of E.

3.2.1. Definition. Let $0 < \alpha = (\alpha_1, ..., \alpha_d) < \infty$ and $1 \leqslant Q = (q_1, ..., q_d) \leqslant \infty$. We define $E_{\alpha;Q;J}$ to be the space of all $x \in E$ for which there is a sequence $(w_M)_{M \in \mathbb{N}^d}$, with $w_M \in W_M$, such that

(1) $$x = \sum_{M \in \mathbb{N}^d} w_M$$

and

$$(e^{\alpha \cdot M} \| w_M \|_E)_{M \in \mathbb{N}^d} \in l^Q(\mathbb{N}^d). \tag{2}$$

The spaces $E_{\alpha;Q;J}$ are Banach spaces under the norm

$$\|x\|_{\alpha;Q;J} = \inf\{\|(e^{\alpha \cdot M} \| w_M \|_E)_{N \in \mathbb{N}^d}\|_{l^Q(\mathbb{N}^d)} \mid x = \Sigma_M w_M\}. \tag{3}$$

3.2.2. PROPOSITION. *For all $x \in W_M$ we have*

$$\|x\|_{\alpha;Q;J} \leqslant e^{\alpha \cdot M} \|x\|_E. \tag{1}$$

Proof. It is enough to observe that $x = \Sigma_{M'} w_{M'}$, with $w_{M'} = 0$ if $M' \neq M$ and $w_M = x$ if $M' = M$, is decomposition of x as in 3.2.1(1).

The inequality 3.2.2(1) is an inequality of Bernstern type and, as before, this permit us to call $E_{\alpha;Q;J}$ an *approximation space.*

3.3. *The Embedding $E_{\alpha;Q;K} \subset E_{\alpha;Q;J}$*

We give a first connection between the spaces $E_{\alpha;Q;K}$ and $E_{\alpha;Q;J}$.

3.3.1. PROPOSITION. *For all $x \in E_{\alpha;Q;K}$ we have*

$$\|x\|_{\alpha;Q;J} \leqslant 2^d \|x\|_{\alpha;Q;K}. \tag{1}$$

Moreover,

$$E_{\alpha;Q;K} \subset E_{\alpha;Q;J}. \tag{2}$$

Proof. Let $x \in E_{\alpha;Q;K}$ and $\varepsilon > 0$. Then, for all $M \in \mathbb{N}^n$ there is a $w'_M \in W_M$ such that

$$\|x - w'_M\|_E \leqslant (1 + \varepsilon)\, \mathscr{E}_M(x).$$

Now, let us set

$$\begin{aligned} w_M &= \Delta_k w'_M && \text{if} \quad M \neq 0 \\ &= 0 && \text{if} \quad M = 0 \end{aligned}$$

(here $\Delta_k w'_M$ stands for the multiple difference of increment $k = (k_1, ..., k_d)$, where $k \in \square$ is chosen so that $M = k \circ M = (k_1 m_1, ..., k_d m_d)$).

We have $w_M \in W_M$ and since $w'_M \to x$, as $|M| \to \infty$, it follows that

$$\sum_{M \in \mathbb{N}^d} w_M = \lim_{|M| \to \infty} w'_M = x. \tag{3}$$

On the other hand, we have

$$\|w_M\|_E \leqslant \sum_{k' \leqslant k} \|w'_{M-k'} - x\|_E \leqslant \sum_{k' \leqslant k} (1+\varepsilon)\,\mathscr{E}_{M-k'}(x) \tag{4}$$
$$\leqslant 2^{n+1}\,\mathscr{E}_{M-1}(x).$$

Hence, from 3.3.1(3) and 3.3.1(4) we obtain

$$\|x\|_{\alpha;Q;J} \leqslant 2^{n+1}\,\|x\|_{\alpha;Q;K},$$

as desired.

Under some additional hypotheses the inclusion of $E_{\alpha;Q;K}$ in $E_{\alpha;Q;J}$ can be reversed. But to do that we shall need some connections between approximation spaces and interpolation spaces.

3.4. *Approximation Spaces and Interpolation Spaces*

As before, let E be a Banach space and $(W_M \mid M \in \mathbb{N}^n)$ a multiple scale of subspaces of E.

3.4.1. DEFINITION. Let F be a subspace of E such that $U_M W_M \subset F$. We shall say that

$$F \in K(\alpha) \quad \text{iff} \quad \mathscr{E}_M(x) \leqslant Ce^{-\alpha\cdot M}\,\|x\|_F,\ \ x \in F; \tag{1}$$

$$F \in J(\alpha) \quad \text{iff} \quad \|x\|_F \leqslant Ce^{\alpha\cdot M}\,\|x\|_E,\ \ x \in W_M. \tag{2}$$

We shall need the following characterizations of the classes $K(\alpha)$ and $J(\alpha)$.

3.4.2. PROPOSITION. *Let F be a subspace of E such that $\bigcup_M W_M \subset F$. Then*

$$F \in K(\alpha) \quad \textit{iff} \quad F \subset E_{\alpha;\infty;K}; \tag{1}$$

$$F \in J(\alpha) \quad \textit{iff} \quad E_{\alpha;1;J} \subset F; \tag{2}$$

and

$$F \in K(\alpha) \cap J(\alpha) \quad \textit{iff} \quad E_{\alpha;1;J} \subset F \subset F_{\alpha;\infty;K}. \tag{3}$$

Proof. The equivalence 3.4.2(1) follows readily from 3.4.1(1).
Now, if $x = \Sigma_M w_M$, with $w_M \in W_M$, we have

$$\|x\|_F \leqslant \Sigma_M \|w_M\|_F \leqslant \Sigma_M e^{\alpha\cdot M}\,\|w_M\|_E,$$

from 3.4.1(2). Therefore $E_{\alpha;1;J} \subset F$. The converse is immediate.

Now, we are ready to state the connections between interpolation spaces and approximation spaces.

3.4.3. PROPOSITION. *Let there be given* $0 < \alpha_0 = (\alpha_0^1, \ldots, \alpha_0^d)$, $\alpha_1 = (\alpha_1^1, \ldots, \alpha_1^d) < \infty$ *and* $0 < \Theta = (\theta_1, \ldots, \theta_d) < 1$. Consider the associated sequence $(\alpha_k = (\alpha_{k1}^1, \ldots, \alpha_{kd}^d) \mid k = (k_1, \ldots, k_d) \in \square)$, *and set* $\alpha = (\alpha_1, \ldots, \alpha_d)$, *where* $\alpha_j = (1 - \theta_j)\alpha_0^j + \theta_j \alpha_1^j$, $j = 1, 2, \ldots, d$. *Let* $(F_k \mid k \in \square)$ *be a family of approximation subspaces of* E *with respect to the multiple scale* $(W_M \mid M = \mathbb{N}^d)$. *Then,*

$$(F_k \mid k \in \square)_{\Theta;Q;K} \subset E_{\alpha;Q;K} \tag{1}$$

if $F_k \in K(\alpha_k)$, $k \in \square$;

$$E_{\alpha;Q;J} \subset (F_k \mid k \in \square)_{\Theta;Q;J} \tag{2}$$

if $F_k \in J(\alpha_k)$, $k \in \square$; *and*

$$E_{\alpha;Q;J} = (F_k \mid k \in \square)_{\Theta;Q} = E_{\alpha;Q;K} \tag{3}$$

if $F_k \in K(\alpha_k) \cap J(\alpha_k)$, $k \in \square$.

Proof. Let $x \in (F_k \mid k \in \square)_{\Theta;Q;K}$ and $x = \Sigma_k x_k$ with $x_k \in F_k$, $k \in \square$. Since $F_k \in K(\alpha_k)$, $k \in \square$, it follows that

$$\mathscr{E}_M(x) \leqslant \sum_{k \in \square} \mathscr{E}_M(x_k) \leqslant \sum_{k \in \square} C_k e^{-\alpha_k \cdot M} \|x_k\|_{F_k}$$
$$\leqslant C e^{-\alpha_0 \cdot M} \sum_{k \in \square} e^{-(\alpha_k - \alpha_0) \cdot M} \|x_k\|_{F_k}.$$

Now, taking the infimum over all the decompositions $\Sigma_k x_k$, it follows that

$$\mathscr{E}_M(x) \leqslant C e^{-\alpha_0 \cdot M} K(e^{-(\alpha_1 - \alpha_0) \circ M}; x).$$

But $\alpha_j - \alpha_0^j = (1 - \theta_j)\alpha_0^j + \theta_j \alpha_1^j - \alpha_0^j = \theta_j(\alpha_1^j - \alpha_0^j)$, $j = 1, 2, \ldots, d$, and thus

$$e^{\alpha \cdot M} \mathscr{E}_M(x) \leqslant C e^{\{(\alpha_1 - \alpha_0) \circ M\} \cdot \Theta} K(e^{-(\alpha_1 - \alpha_0) \circ M}; x). \tag{4}$$

If we take the $l^Q(\mathbb{N}^d)$ norm on both sides, a standard argument yields the embedding 3.4.3(1).

Let $x \in E_{\alpha;Q;J}$ and consider a decomposition $x = \Sigma_M w_M$, with $w_M \in W_M$ and $M \in \mathbb{N}^d$. Then, setting $u_M = w_{-M}$ when $M \leqslant 0$ and $u_M = 0$ elsewhere, we have

$$x = \sum_{M \in \mathbb{Z}^d} u_M.$$

Since $F_k \in J(\alpha_k)$, $k \in \square$, we have

$$J(e^{(\alpha_1 - \alpha_0) \circ M}; u_M) = \max_k e^{(\alpha_k - \alpha_0) \cdot M} \|w_{-M}\|_{F_k} \leqslant C e^{-\alpha_0 \cdot M} \|w_{-M}\|_E.$$

Hence

$$e^{-\{(\alpha_1-\alpha_0)\circ M\}\cdot\Theta} J(e^{(\alpha_1-\alpha_0)\circ M}; u_M) \leqslant Ce^{-\alpha\cdot M} \| w_{-M} \|_E.$$

If we take the l^Q-norm, embedding 3.4.3(2) follows.

Finally, identities 3.4.3(3) follows at once from 3.3.1(2), 3.4.3(1), 3.4.3(2) and the identity between the K- and J-interpolation spaces. This completes the proof.

4. Approximation by Entire Functions of Exponential Type

We shall now apply the d-parametric approximation theory of Section 3 to establish direct and converse approximation theorems by entire functions of exponential type in functions spaces with a dominant mixed derivative. In this way, we shall obtain some results by Nikol'skiĭ (see for instance Nikol'skiĭ [13]) as consequences of the abstract approximation theory.

4.1. *The Multiple Scale of Entire Functions of Exponential Type*

4.1.1. Let there be given a vector $R=(r_1,..., r_d) \geqslant 0$ with integral components. We shall consider entire functions of exponential type $\leqslant R = (r_1,..., r_d)$, i.e., entire functions such that for each given $\varepsilon > 0$ there exists an $M(\varepsilon) > 0$ for which we have

$$| g(Z)| \leqslant M(\varepsilon) \exp \left\{ \sum_{j=1}^{d} (r_j+\varepsilon) \,| z_j | \right\}$$

for all $Z=(z_1,..., z_d) \in \mathbb{C}^d$.

4.1.2. We take E to be $L^P = L^P(\mathbb{R}^d)$, $1 \leqslant P = (p_1,..., p_d) \leqslant \infty$, the L^P space with mixed norm of Benedek and Panzone [2].

The subspace W_M will be defined as the space of all functions in L^P that are entire of exponential type $\leqslant M \circ R = (m_1 r_1,..., m_d r_d)$, and will be denoted by $I^{M\circ R,P}$. Thus, by $f \in I^{M\circ R,P}$ we shall mean that the Fourier transform $\hat{f}$ vanishes outside the set $|t_j| \leqslant e^{m_j r_j}$, $j = 1, 2,..., d$.

We see that $(I^{M\circ R,P} \mid M \in \mathbb{N}^d)$ is a multiple scale of subspaces of L^P.

4.2. *Characterization of the Approximation Space* $[L^P]_{\alpha,Q}$

Let $(I^{N\circ R,P} \mid N \in \mathbb{N}^d)$ be the multiple scale introduced in 4.1.

4.2.1. Proposition. *The space $W^{M,P}(\mathbb{R}^d)$ is of class $J(\alpha)$, where $\alpha = (\alpha_1,..., \alpha_d) = (m_r r_1,..., m_d r_d)$, i.e.,*

(1) $$\|f\|_{W^{M,P}} \leqslant Ce^{N\circ\alpha} \|f\|_{L^P},$$

for all $f \in I^{N\circ\alpha,P}(\mathbb{R}^d)$.

Proof. To prove that $W^{M,P}$ is of class $J(\alpha)$ it is evidently enough to verify the inequalities

$$\|D^{k\circ M}f\|_{L^P} \leqslant Ce^{k\circ N\circ\alpha}\|f\|_{L^P} \qquad (k\in\square)$$

for all $f\in I^{N\circ\alpha,P}$. These inequalities are immediate consequences of

$$\|D^{k\circ M}f\|_{L^P} \leqslant C\|f\|_{L^P},$$

where f is of exponential type $\leqslant 1=(1,\dots,1)$. Indeed, if $f\in I^{N\circ\alpha,P}$, then the function

$$g(x)=f(e^{-N\circ\alpha}\circ x)$$

is of exponential type $\leqslant 1=(1,\dots,1)$. Inserting this function in (3) we easily get (2).

To prove (3), let $f\in I^{1,P}$, so that the Fourier transform $\hat{f}$ has support in $\{|t_j|\leqslant 1, j=1,2,\dots,d\}$. Now, if $\psi\in C_c^\infty(\mathbb{R}^d)$ equals 1 in a neighborhood of the set $\{|t_j|\leqslant 1, j=1,2,\dots,d\}$, we have

$$\hat{f}=\psi\hat{f}=\widehat{\tilde{\psi}*f},$$

where $\tilde{h}$ denotes the inverse Fourier transform of h. Hence,

$$f=\tilde{\psi}*f,$$

and consequently

$$D^{k\circ M}f=D^{k\circ M}(\tilde{\psi}*f)=(D^{k\circ M}\tilde{\psi})*f.$$

Since $D^{k\circ M}\tilde{\psi}\in S(\mathbb{R}^d)$, Young's inequality yields

$$\|D^{k\circ M}f\|_{L^P}\leqslant\|D^{k\circ M}\tilde{\psi}\|_{L^1}\|f\|_{L^P}.$$

Finally, since ψ is taken independently of f, if we put $C=\max\{\|D^{k\circ M}\tilde{\psi}\|_{L^1}\mid k\in\square\}$ the inequality 3.1(3) follows.

4.2.2. Proposition. *The space $W^{M,P}(\mathbb{R}^d)$ is of class $K(\alpha;L^P)$ where $\alpha=(\alpha_1,\dots,\alpha_d)=(m_1r_1,\dots,m_dr_d)$, i.e.,*

$$(1) \qquad \mathscr{E}_N(f)\leqslant Ce^{-N\circ\alpha}\|f\|_{W^{M,P}}$$

for all $f\in W^{M,P}(\mathbb{R}^d)$.

Proof. To prove 4.2.2(1) it is enough to show that given $f\in W^{M,P}$, there exists $w\in I^{N\circ R,P}$ such that

$$(2) \qquad \|f-w\|_{L^P}\leqslant Ce^{-N\circ\alpha}\|f\|_{W^{M,P}},$$

which obviously is a consequence of

$$\|f - w\|_{L^P} \leqslant Ce^{-N \circ \alpha} \|D^M f\|_{L^P},$$

which in turn is a consequence of

$$\|f - w\|_{L^P} \leqslant C \|D^M f\|_{L^P},$$

where now $w \in I^{1,P}$.

To prove 4.2.2(1) let $\hat{p}$ be a $C_c(\mathbb{R}^d)$-function that vanishes outside of the set $|t_1| \leqslant 1$ and equals 1 in $|t_j| \leqslant 1/2$, $j = 1, 2, \ldots, d$. Define w by the formula

$$\hat{w} = \hat{p}\hat{f}.$$

Hence

$$\hat{f}(t) - \hat{w}(t) = \{1 - \hat{p}(t)\} \hat{f}(t) = \{1 - \hat{p}(t)\}(it)^{-M}(it)^M \hat{f}(t),$$

and putting $\hat{\psi}(t) = \{1 - \hat{p}(t)\}(it)^{-M}$ we shall have

$$f - w = \psi * D^M f,$$

Finally, since $\psi \in L^1$ and $D^M f \in L^P$, Young's inequality yields

$$\|f - w\|_{L^P} \leqslant \|\psi\|_{L^1} \|D^M f\|_{L^P}.$$

Since ψ is fixed independently of f, we take $C = \|\psi\|_{L^1}$ and 4.2.1(4) follows.

We are now ready to characterize the approximation space $[L^P]_{\alpha,Q}$.

4.2.3. Proposition. *Let us consider the multiple scale* $(I^{N \circ R} \mid N \in \mathbb{N}^d)$. *Fix* $M = (m_1, \ldots, m_d) \in \mathbb{N}^d$ *and* $0 < \Theta = (\theta_1, \ldots, \theta_d) < 1$, *and define* $\alpha_k = (\alpha^1_{k_1}, \ldots, \alpha^d_{k_d})$ *by* $\alpha^j_{k_j} = (m_j + k_j) r_j$, $j = 1, 2, \ldots, d$, *and* $k = (k_1, \ldots, k_d) \in \square$. *Now, if* $\alpha = (\alpha_1, \ldots, \alpha_d)$ *is given by* $\alpha_j = (1 - \theta_j)\alpha^j_0 + \theta_j \alpha^j_1$, $j = 1, 2, \ldots, d$, *and* $1 \leqslant Q = (q_1, \ldots, q_d) \leqslant \infty$, *we have*

$$[L^P]_{\alpha,Q} = B_P^{M+\Theta,Q}. \tag{1}$$

Proof. Since $\alpha_k = (M + k) \circ R$, for all $k \in \square$, we have

$$W^{M+k,P} \in K(\alpha_k) \cap J(\alpha_k).$$

Hence

$$(W^{N+k,P} \mid k \in \square)_{\Theta;Q} = [L^P]_{\alpha,Q}.$$

On the other hand, by 2.2.5(1) we have

$$(W^{M+k,P} \mid k \in \square)_{\Theta;Q} = B_P^{M+\Theta,Q},$$

as desired.

4.2.4. *Remark.* Another characterization of the spaces $[L^P]_{\alpha,Q}$ is given in Fernandez [7]. Let $(\varphi_N)_N$ be a system of test functions (see [7] or [17] for the definition), $S=(s_1,..., s_d)\in \mathbb{R}^d$, $1\leqslant P=(p_1,..., p_n)$, $Q=(q_1,..., q_n)\leqslant \infty$. We set

$$B_Q^{SP}=B_Q^{SP}(\mathbb{R}^n)=\{f\in S'(\mathbb{R}^n) \mid (\varphi_N * f)_{N\in \mathbb{N}^n}\in l^Q(L^P)\}.$$

It is shown in [7] that these spaces B_Q^{SP} and the spaces B_Q^{SP}, considered in this article coincide when $S=(s_1,..., s_d)>0$. On the other hand, this definition of the spaces B_Q^{SP} should be compared with the iterative definition given by Sparr [17, p. 300]. Although we have no counterexample it seems that our spaces do not coincide with Sparr' spaces. This may give rise to the question as to whether the approximation spaces studied here may be obtained in an iterative way. As the spaces B_Q^{SP} do not seem to be iterative the same ought to be true for the abstract approximation spaces. This and other related matters will be treated in a forthcoming paper [8], now in preparation.

4.3. *Direct and Converse Approximation Theorems*

As a consequence of the foregoing results we shall state direct and converse approximation theorems of Jackson and Bernstein type.

4.3.1. PROPOSITION (Theorem of Jackson type). *If $f\in B_P^{S,Q}$ then*

$$\mathscr{E}_N(f)=O(e^{-N\circ S}) \tag{1}$$

for all $N\in \mathbb{N}^d$.

Proof. Fix $N=(n_1,..., n_d)$, and choose $M=(m_1,..., m_d)$ and $0<\Theta=(\theta_1,..., \theta_d)$ such that $0\leqslant N\leqslant M-1$ and $S=N+\Theta$. If we take $R=(1,..., 1)$ in Proposition 4.2.3, it follows that

$$[L^P]_{S,Q}=B_P^{S,Q}.$$

Finally, by the inequality of Jackson type 3.1.4(1) it follows that

$$\mathscr{E}_N(f)\leqslant e^{-N\circ S}\,\|f\|_{B_P^{S,Q}},$$

and hence the desired result.

4.3.2. PROPOSITION (Theorem of Bernstein type). *Let $f\in L^P(\mathbb{R}^d)$ such that*

$$\mathscr{E}_N(f)=O(e^{-N\circ S}). \tag{1}$$

Then

$$f\in B_P^{S,\infty}. \tag{2}$$

Proof. Let $M = (m_1, \ldots, m_d)$ and $0 < \Theta = (\theta_1, \ldots, \theta_d) < 1$ be chosen such that $S = M + \Theta$. Then 4.3.2(1) implies that

$$[L^P]_{S,\infty} = (W^{M+k,P} \mid k \in \square)_{\Theta;\infty},$$

and thus 4.3.2(1) yields 4.3.2(2) at once.

The next two propositions generalize the preceding results.

4.3.3. Proposition. *Let* $f \in B_P^{S,Q}$ *and* $S = (s_1, \ldots, s_d) > 1$. *Then, if* $S = M + \alpha$ *with* $0 < \alpha = (\alpha_1, \ldots, \alpha_d) < 1$, *it follows that, for all* $k \in \square$,

$$\mathscr{E}_N(D^{M-k}f) = O(e^{-N \circ \alpha}). \tag{1}$$

Proof. By the reduction theorem the hypothesis $f \in B_P^{S,Q}$ implies that $f \in W^{M,P}$ and, for all $k \in \square$, there holds

$$D^{M-k}f \in B_P^{\alpha,Q}.$$

On the other hand, $B_P^{\alpha,Q} = [L^P]_{\alpha,Q}$, and so by the inequality of Jackson type 3.1.4(1) it follows that

$$\mathscr{E}_N(D^{M-k}f) \leqslant C e^{-N \circ \alpha} \|f\|_{B_P^{\alpha,\infty}}.$$

This proves our contention.

4.3.4. Proposition. *Let* $f \in L^P$, *and if* $S = (s_1, \ldots, s_d) > 1$ *suppose that*

$$\mathscr{E}_N(f) = O(e^{-N \circ S}). \tag{1}$$

Then, if $S = M + \alpha$ *with* $M = (m_1, \ldots, m_d) \in \mathbb{N}^d$ *and* $0 < \alpha = (\alpha_1, \ldots, \alpha_d) < 1$, *we have*

$$f \in W^{M,P}, \tag{2}$$

and, for all $k \in \square$,

$$D^{M-k}f \in B_P^{\alpha,\infty}. \tag{3}$$

Proof. From 4.3.4(1) we see that $f \in [L^P]_{S,\infty} = B_P^{S,\infty}$. The result follows at once from the reduction theorem.

5. On the Representation by Entire Functions of Exponential Type

We close this article by showing that a theorem by Nikol'skiĭ on the representation of functions in $B_P^{S,Q}$ by sums of series of entire functions of exponential type (see Nikol'skiĭ [13] and Amanov [1]) is also a consequence of our J-approximation theory.

To avoid notational complications we shall restrict ourselves to the case $d=2$. Also, we shall consider the double scale $\{I^{mn} \mid m, n \geqslant 0\}$, i.e., the multiple scale described in 4.1.1 in the case $d=2$ and with $R=(1,1)$.

5.0.1. PROPOSITION. *If $f \in B_P^{S,Q}$, then there exists a double sequence $(w_{mn})_{m\geqslant 0, n\geqslant 0}$ with $w_{mn} \in W_{mn}$, such that*

$$f = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} w_{mn} \qquad (\text{in } B_P^{S,Q}). \tag{1}$$

Proof. This follows immediately from the fact that $B_P^{S,Q} = [L^P]S_{\alpha,Q;J}$.

REFERENCES

1. T. I. AMANOV, Representation and imbedding theorems for the function space $S^r_{p,\theta}B(\mathbb{R}_n)$, *Proc. Steklov Inst. Math.* **77** (1965), 3–36.
2. A. BENEDEK AND R. PANZONE, The space L^P, with mixed norm, *Duke Math. J.* **28** (1961), 301–324.
3. J. BERG AND J. LÖFSTRÖM, "Interpolation Spaces," Springer-Verlag, Berlin/Heidelberg, 1976.
4. P. L. BUTZER, A survey of work on approximation at Aachen, 1968–1972, *in* "Approximation Theory, Proceedings, International Symposia," pp. 31–100, Academic Press, New York, 1973.
5. D. L. FERNANDEZ, Interpolation of 2^n Banach spaces, *Studia Math.* **65** (1979), 175–201.
6. D. L. FERNANDEZ, Interpolation of Sobolev–Nikol'skiĭ spaces, to appear.
7. D. L. FERNANDEZ, On Besov–Nikol'skiĭ spaces, to appear.
8. D. L. FERNANDEZ, On the interpolation of 2^n Banach spaces, II, in preparation.
9. H. JOHNEN, Über Sätze von M. Zamansky und S. B. Steckin und ihre Umkehrungen auf der n-dimensionale Torus, *J. Approx. Theory* **2** (1969), 97–110.
10. J. L. LIONS AND J. PEETRE, Sur une classe d'espaces d'interpolation, *Publ. Math. IHES* **19** (1964), 5–68.
11. P. I. LIZORKIN, Multipliers of Fourier integrals and bounds of convolution in spaces with mixed norms. Applications, *Math. Izv.* **4** (1970), 225–255.
12. P. I. LOZORKIN AND S. M. NIKOL'SKII, A classification of differentiable functions in some fundamental spaces with dominant mixed derivatives, *Proc. Steklov Inst. Math.* **77** (1965), 160–187.
13. S. M. NIKOL'SKIĬ, Functions with a dominant mixed derivatives satisfaying a multiple Hölder condition, *Amer. Math. Soc. Transl.* (2) **102** (1973), 27–51.
14. S. M. NIKOL'SKIĬ, "Approximation of Functions of Several Variables and Imbedding Theorems," Springer-Verlag, Berlin/Heidelberg, 1975.
15. J. PEETRE, Espaces d'inionterpolation et théorème de Soboleff, *Ann. Inst. Fourier* **16** (1966), 279–317.

16. J. PEETRE, "A Theory of Interpolation of Normed Spaces," Notas de Matemática No. 39, IMPA, Rio de Janeiro, 1968.
17. G. SPARR, Interpolation of several Banach spaces, *Ann. Mat. Pura Appl.* **99** (1974), 241–316.
18. A. YOSHIKAWA, Sur la théorie d'espaces d'interpolation—les espaces de moyenne de plusieurs espaces de Banach, *J. Fac. Sci. Univ. Tokyo* **16** (1970), 407–468.